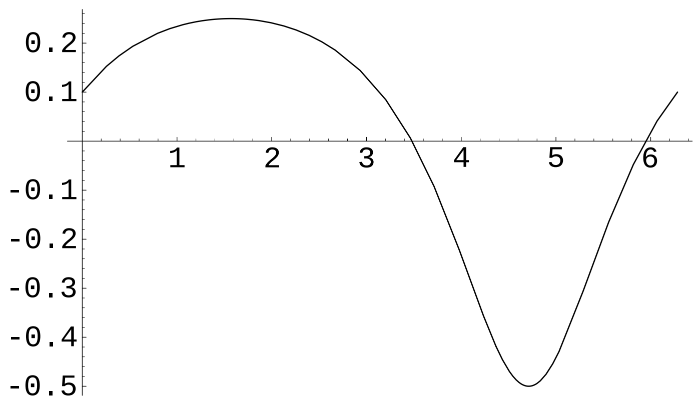
In one approach to Problem 3 on Project 3, you come up with the integral

$$\int_0^{2\pi} \frac{1+3\sin\theta}{10+6\sin\theta}\,d\theta.$$

Here's a plot of the integrand:



If you use technology (such as a calculator or *Mathematica*) to evaluate this definite integral, you get the correct result of 0. (I'm not sure what a TI-89 will return for an antiderivative.)

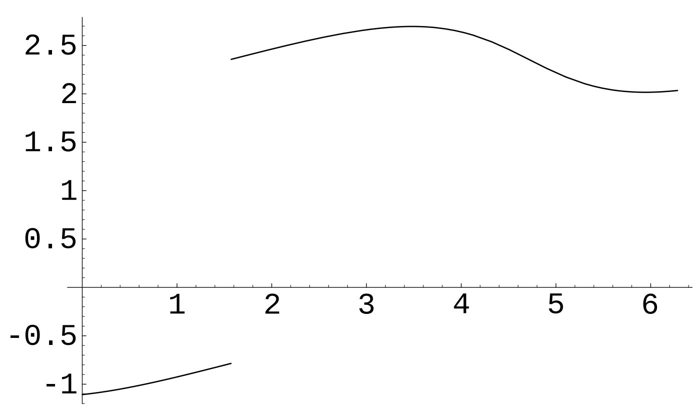
On the other hand, if you ask *Mathematica* for an antiderivative of the integrand, you get

$$F(\theta) = \frac{\theta}{2} - \arctan\left(\frac{2\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}\right)$$

With this, you can calculate

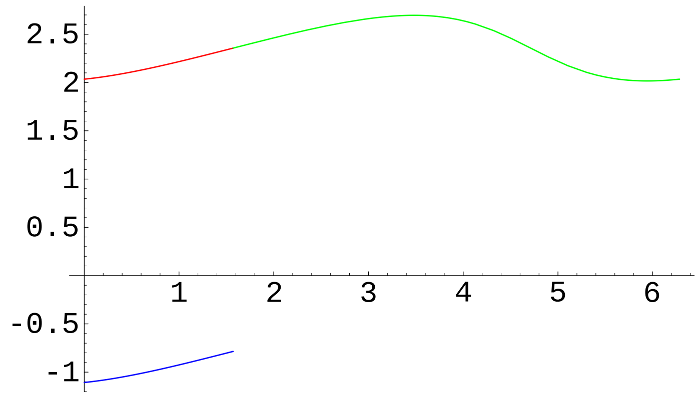
$$\begin{aligned} F(2\pi) - F(0) &= \left[\pi - \arctan\left(\frac{2\left(\cos\pi + \sin\pi\right)}{\cos\pi - \sin\pi}\right)\right] - \left[0 - \arctan\left(\frac{2\left(\cos 0 + \sin 0\right)}{\cos 0 - \sin 0}\right)\right] \\ &= \pi - \arctan(2) - 0 + \arctan(2) \\ &= \pi \end{aligned}$$

What's going on here? Why don't we get the correct result of 0? The answer is connected to the fact that we need to be careful about differentiablilty. Here's a plot of the function $F(\theta)$:



This function is not continuous at $\theta = \pi/2$. (You might have noticed from the formula for $F(\theta)$ that this value might be a problem since $\cos(\pi/4) - \sin(\pi/4) = 0$.) So, the function that *Mathematica* returned is not continuous throughout the interval $[0, 2\pi]$ and thus is not differentiable throughout that interval. Thus, this function is not an antiderivative that we can use in the Second Fundamental Theorem of Calculus.

Can we find a antiderivative that is valid for the interval $[0, 2\pi]$? Yes, by using a piecewise defined function. Note that if we raise the piece of the graph to the left of $\theta = \pi/2$ by the correct amount, it will connect up with the piece to the right, as shown in the following plot.



In the above plot, the blue and green curves are the graph of the original function given by *Mathematica*. The red curve is a translation of the blue curve by π units in the vertical direction. The red and green curves give the graph of the antiderivitive we are after. A formula for this function is

$$\begin{aligned}\tilde{F}(\theta) &= \begin{cases} F(\theta) + \pi & \text{if } 0 \le \theta \le \frac{\pi}{2} \\ F(\theta) & \text{if } \frac{\pi}{2} \le \theta \le 2\pi \end{cases} \\ &= \begin{cases} \frac{\theta}{2} - \arctan\left(\frac{2\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}\right) + \pi & \text{if } 0 \le \theta \le \frac{\pi}{2} \\ \frac{\theta}{2} - \arctan\left(\frac{2\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}\right) & \text{if } \frac{\pi}{2} \le \theta \le 2\pi \end{cases}\end{aligned}$$

You can confirm that $\tilde{F}(2\pi) - \tilde{F}(0) = 0$. (To be complete, we really should prove that $\tilde{F}$ is differentiable at $\theta = \pi/2$.)